Trento 8 Giugno 2016

The Wheeler-DeWitt Equation in Distorted Gravity

Remo Garattini
Università di Bergamo
I.N.F.N. - Sezione di Milano

Plan of the Talk

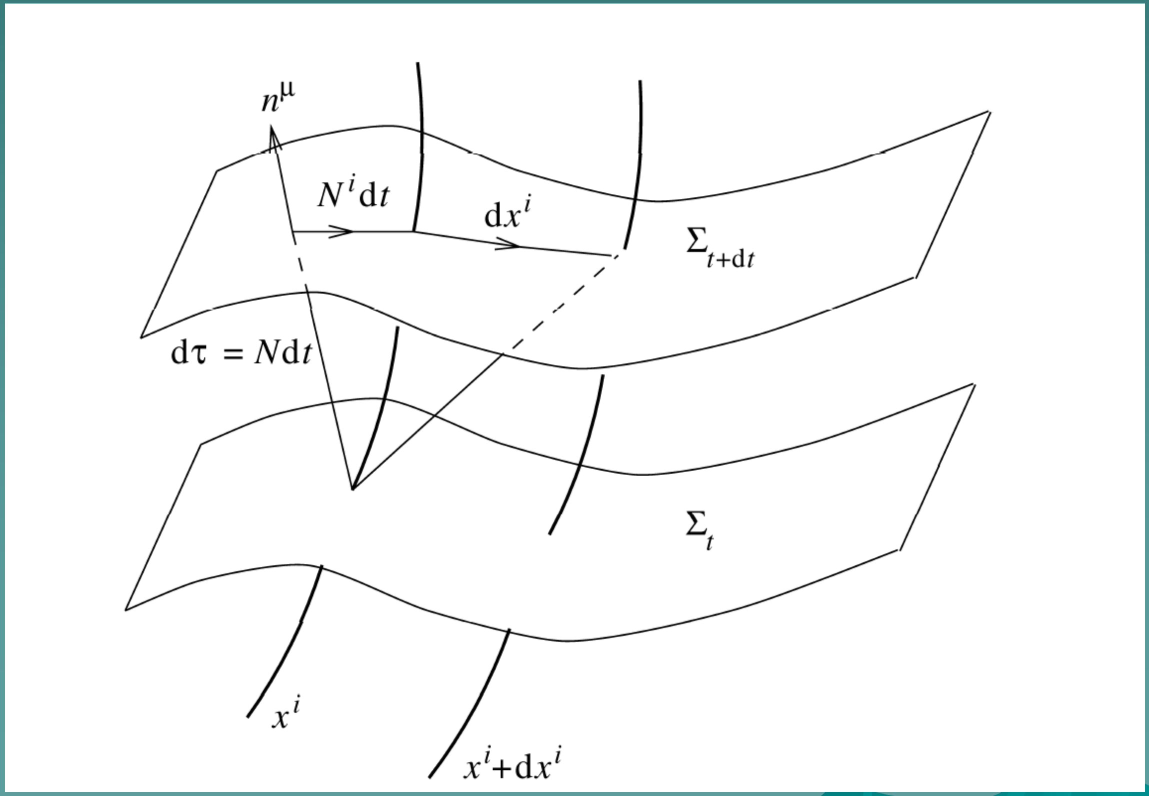
- Building the Wheeler-DeWitt Equation
- The Wheeler-DeWitt Equation as a Sturm-Liouville problem
- Distorting Gravity in a MSS approach for a FLRW model.
- The Cosmological Constant as a Zero Point Energy Computation in the Gravity's Rainbow context
- Conclusions and Outlooks

Relevant Action for Quantum Cosmology

$$S=\frac{1}{2\kappa}\int_{\mathfrak{M}} d^4x\sqrt{-g}\left({}^4R-2\Lambda\right)+2\int_{\partial\mathfrak{M}} d^3x\sqrt{g^{(3)}}K+S_{matter} \qquad \kappa=8\pi G$$

$G\rightarrow$ Newton's Constant

$\Lambda\rightarrow$ Cosmological Constant



Relevant Action for Quantum Cosmology

$$S=\frac{1}{2\kappa}\int_{\mathfrak{M}} d^4x\sqrt{-g}\left({}^4R-2\Lambda\right)+\frac{1}{\kappa}\int_{\partial\mathfrak{M}} d^3x\sqrt{g^{(3)}}K+S_{matter}$$

ADM Decomposition

$$g_{\mu\nu}=\begin{pmatrix}-N^2+N^kN_k & N_j\\ N_i & g_{ij}^{(3)}\end{pmatrix}\qquad g^{\mu\nu}=\begin{pmatrix}-\frac{1}{N^2} & \frac{N^j}{N^2}\\ \frac{N^i}{N^2} & g^{ij(3)}-\frac{N^iN^j}{N^2}\end{pmatrix}$$

$$\mathrm{ds}^2=g_{\mu\nu}dx^\mu dx^\nu=-N^2dt^2+g_{ij}\left(N^idt+dx^i\right)\left(N^jdt+dx^j\right)$$

$$K_{ij}=-\frac{1}{2N}\dot{g}_{ij}+\nabla_iN_j+\nabla_jN_i$$

N is the lapse function N_i is the shift function

$$K=K^{ij}g_{ij}$$

$$S=\frac{1}{2\kappa}\int_{\Sigma\times I} dtd^3xN\sqrt{g^{(3)}}\left(K^{ij}K_{ij}-K^2+{}^3R-2\Lambda\right)+S_{\partial(\Sigma\times I)}+S_{matter}$$

Legendre Transformation $\rightarrow$ $H=\int_\Sigma d^3x\left(N_i\mathcal{H}^i+N\mathcal{H}\right)+H_{\partial\Sigma}$

$$\mathcal{H}=(2\kappa)G_{ijkl}\pi^{ij}\pi^{kl}-\frac{\sqrt{g}}{2\kappa}\left({}^3R-2\Lambda\right)=0$$

Classical Constraint $\rightarrow$ Invariance by time reparametrization

$\mathcal{H}^i=2\pi^{ij}{}_{|j}=0$ Classical Constraint $\rightarrow$ Gauss Law

Wheeler-De Witt Equation

B. S. DeWitt, Phys. Rev.**160**, 1113 (1967).

$$\left[(2\kappa)G_{ijkl}\pi^{ij}\pi^{kl}-\frac{\sqrt{g}}{2\kappa}(R-2\Lambda)\right]\Psi\left[g_{ij}\right]=0$$

- G_{ijkl} is the super-metric,
- R is the scalar curvature in 3-dim.

Example:WDW for Tunneling

$$ds^2=-N^2dt^2+a^2(t)d\Omega_3^2$$

$$H\Psi[a]=\left[-\frac{\partial^2}{\partial a^2}-\frac{q}{a}\frac{\partial}{\partial a}+\frac{9\pi^2}{4G^2}\left(a^2-\frac{\Lambda}{3}a^4\right)\right]\Psi[a]=0$$

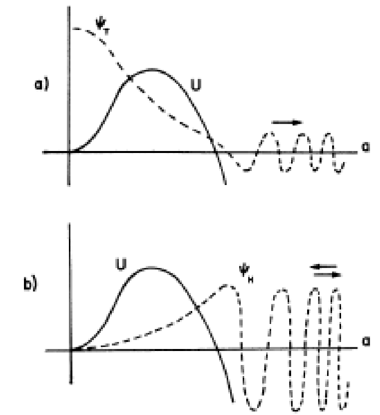


FIG. 2. (a) Tunneling and (b) Hartle-Hawking wave functions for the one-dimensional minisuperspace model describing a de Sitter space. The "potential" $U(a)$ is shown by a solid line and the wave functions by dashed lines.

Formal Schrödinger Equation with zero eigenvalue whose solution is a linear combination of Airy's functions (q=-1 Vilenkin Phys. Rev. D 37, 888 (1988).) containing expanding solutions

Wheeler-De Witt Equation

B. S. DeWitt, Phys. Rev.**160**, 1113 (1967).

$$H\Psi[a]=\left[-\frac{1}{a^q}\frac{\partial}{\partial a}\left(a^q\frac{\partial}{\partial a}\right)+\frac{9\pi^2}{4G^2}\left(a^2-\frac{\Lambda}{3}a^4\right)\right]\Psi[a]=E\Psi[a]\Leftrightarrow E=0$$

E=0 is highly degenerate ⟺ *Sturm-Liouville Eigenvalue Problem*

→ *Variational procedure*

$$\left[\frac{d}{dx}\left(p(x)\frac{d}{dx}\right)+q(x)+\lambda w(x)\right]y(x)=0$$

$$\int_a^b w(x)\,y^*(x)\,y(x)\,dx\leftrightarrow \text{Normalization with weight } w(x)\rightarrow\int_0^\infty a^{q+4}\Psi^*(a)\Psi(a)\,da$$

$$p(x)\rightarrow a^q(t)\quad q(x)\rightarrow-\left(\frac{3\pi}{2G}\right)^2 a^{q+2}(t)\quad w(x)\rightarrow a^{q+4}(t)\quad y(x)\rightarrow\Psi[a]\quad \lambda\rightarrow\left(\frac{3\pi}{2G}\right)^2\left(\frac{\Lambda}{3}\right)$$

$$\lambda = \min_{y(x)} \frac{-\int_a^b y^*(x)\left[\frac{d}{dx}\left(p(x)\frac{d}{dx}\right)+q(x)\right]y(x)\,dx}{\int_a^b w(x)\,y^*(x)\,y(x)\,dx} \to$$

Rayleigh-Ritz
Variational Procedure

$$y(a)=y(b)=0$$

$$\frac{\Lambda}{3}\left(\frac{3\pi}{2G}\right)^2 = \min_{\Psi(a)} \frac{\int_0^\infty \Psi^*(a)\left[-\frac{d}{da}\left(a^q\frac{d}{da}\right)+\left(\frac{3\pi}{2G}\right)^2 a^{q+2}\right]\Psi(a)\,da}{\int_0^\infty a^{q+4}\Psi^*(a)\Psi(a)\,da}$$

$\Psi(0)\neq 0$ for example for q=0

$\Psi(a)=\exp\left(-\beta a^2\right)\to$ No Solution

$\Psi(\infty)=0$

$\Psi(0)=0\leftarrow$ De Witt Condition

Distorting Gravity

Hořava-Lifshitz theory → UV Completion, problems with scalar graviton in IR

Varying Speed of Light Cosmology→ Solve problems in the Inflationary phase (horizon,flatness, particle production)

Gravity's Rainbow→ Like VSL. Moreover it allows finite calculation to one loop. The set of the Rainbow's functions is too large. A selection procedure is necessary

G.U.P.→The usual Heisenberg U.P. is modified at very high energies (Planck?!?)

We can include also Noncommutative geometries, f(R) theories….

At low energy all these models describe GR

Gravity's Rainbow

Doubly Special Relativity

G. Amelino-Camelia, Int.J.Mod.Phys. D 11, 35 (2002); gr-qc/001205.
G. Amelino-Camelia, Phys.Lett. B 510, 255 (2001); hep-th/0012238.

$$E^2 g_1^2\left(E/E_P\right) - p^2 g_2^2\left(E/E_P\right) = m^2$$

$$\lim_{E/E_P \to 0} g_1\left(E/E_P\right) = \lim_{E/E_P \to 0} g_2\left(E/E_P\right) = 1$$

Curved Space Proposal → *Gravity's Rainbow*
[J. Magueijo and L. Smolin, Class. Quant. Grav. 21, 1725 (2004) arXiv:gr-qc/0305055].

$$ds^2 = -\frac{N(r)\,dt^2}{g_1^2\left(E/E_P\right)} + \frac{dr^2}{\left(1-\frac{b(r)}{r}\right) g_2^2\left(E/E_P\right)} + \frac{r^2}{g_2^2\left(E/E_P\right)} d\theta^2 + \frac{r^2}{g_2^2\left(E/E_P\right)} \sin^2\theta\, d\varphi^2$$

$N(r) = \exp\left(-2\Phi(r)\right)$ $\Phi(r)$ is the redshift function

$b(r)$ is the shape function Condition $\to b(r_0) = r_0 \quad r \in \left[r_0, +\infty\right)$

Gravity's Rainbow $\Longrightarrow$ Application to Hořava-Lifshitz theory

[R. Garattini and E.N.Saridakis, Eur.Phys.J. C75 (2015) 7, 343; arXiv:1411.7257 [gr-qc]]

$$ds^2 = -N^2(t)\,dt^2 + a^2(t)\,d\Omega_3^2 \quad \Leftrightarrow \text{ FLRW metric}$$

$$\mathcal{L}_{Pp} = N\sqrt{g}\Big\{ g_0\kappa^{-1} + g_1 R + \kappa\left(g_2 R^2 + g_3 R^{ij}R_{ij}\right) + \kappa^2\left(g_4 R^3 + g_5 R R^{ij}R_{ij} + g_6 R^i_j R^j_k R^k_i + g_7 R\nabla^2 R + g_8 \nabla_i R_{jk}\nabla^i R^{jk}\right)\Big\},$$

$$\mathcal{L}_P = N\sqrt{g}\left[g_0\kappa^{-1} + g_1\frac{6}{a^2(t)} + \frac{12\kappa}{a^4(t)}(3g_2+g_3) + \frac{24\kappa^2}{a^6(t)}(9g_4+3g_5+g_6)\right].$$

$g_0\kappa^{-1} = 2\Lambda \quad g_1 = -1 \quad \begin{cases} 3g_2 + g_3 = b \\ 9g_4 + 3g_5 + g_6 = c \end{cases} \quad b = c = 0 \rightarrow GR$

$$ds^2 = -\frac{N^2(t)}{g_1^2(E/Ep)}dt^2 + \frac{a^2(t)}{g_2^2(E/Ep)}d\Omega_3^2 \quad \Leftrightarrow \text{ Distorted } \textit{FLRW metric}$$

Gravity's Rainbow $\Longrightarrow$ Application to Hořava-Lifshitz theory

[R. Garattini and E.N.Saridakis, Eur.Phys.J. C75 (2015) 7, 343; arXiv:1411.7257 [gr-qc]]

$$\left[-\frac{\partial^2}{\partial a^2}-\frac{q}{a}\frac{\partial^2}{\partial a}+\left(\frac{3\pi g_2\left(E/E_P\right)}{2Gg_1\left(E/E_P\right)}\right)^2 a^2\left(1-\frac{\Lambda a^2}{3g_2^2\left(E/E_P\right)}\right)\right]\Psi(a)=0$$

$$If \quad E\equiv E\left(a(t)\right)$$

FIXING

$$\begin{cases} g_1^2\left(E/E_P\right) f\left(a(t),a\right)=1 \\ g_2^2\left(E/E_P\right)=1-c_1\dfrac{E^2\left(a(t)\right)}{E_P^2}-c_2\dfrac{E^4\left(a(t)\right)}{E_P^4} \end{cases}$$

$$f\left(a(t),a\right)=1-2a(t)A(t)+A^2(t)a^2(t)$$

$$A(t)=\frac{1}{g_2\left(E\left(a(t)\right)/E_P\right)E_P}\frac{dg_2\left(E\left(a(t)\right)/E_P\right)}{dE}\frac{dE}{da}$$

Gravity's Rainbow $\Longrightarrow$ Application to Hořava-Lifshitz theory

[R. Garattini and E.N.Saridakis, Eur.Phys.J. C75 (2015) 7, 343; arXiv:1411.7257 [gr-qc]]

then using the "normal" dispersion relation $E^2 = \dfrac{k^2}{a^2(t)}$ and $E_P^2 = \dfrac{k^2}{a_P^2} = \dfrac{k^2}{l_P^2} = \dfrac{k^2}{G}$

$$g_2^2(E/E_P) = 1 - \frac{16b\pi G}{a^2(t)} - \frac{256\pi^2 G^2}{a^4(t)} = 1 - \frac{16b\pi R}{R_0} - \frac{256\pi^2 R^2}{R_0^2} \leftarrow$$

Potential part of the Projectable Hořava-Lifshitz theory without detailed balanced Condition z=3

$$E_P^2 = G^{-1}, \qquad c_1 = 16b\pi \qquad \text{and} \qquad c_2 = 256c\pi^2.$$

It is possible to build a map also for a SSM

Applying the Rayleigh-Ritz procedure we can find candidate eigenvalues depending on the combination of the coupling constants
[R. G., P.R.D 86 123507 (2012) 7, 343; arXiv:0912.0136 [gr-qc]]

A Brief Mention to GUP

[R. Garattini and Mir Faizal; **N.P. B 905 (2016) 313** arXiv:1510.04423 [gr-qc]]

Deformed Momentum

$$\pi_a = \tilde{\pi}_a (1 - \alpha ||\tilde{\pi}_a|| + 2\alpha^2 ||\tilde{\pi}_a||^2)$$

Deformed U.P.

$$\Delta a \Delta \pi_a = 1 - 2\alpha < \pi_a > +4\alpha^2 < \pi_a^2 >$$

Trial Wave Function

$$\Psi(x) = x^\beta \exp\left(-\frac{\beta x^4}{2}\right)$$

Flat space

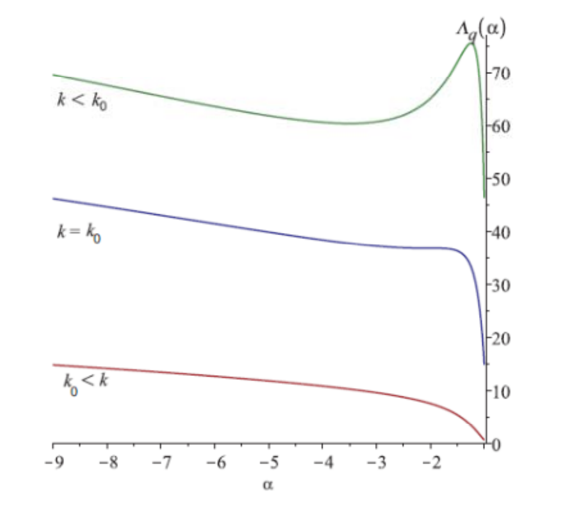
α_0	β_m	$\tilde{\Lambda}_{\alpha_0}(\beta_m)$
1	1.053	29.24
10	1.017	246.29
20	1.012	481.46

A Brief Mention to VSL

[R. Garattini and R.G. and M.De Laurentis, arXiv:1503.03677]

$$ds^2 = -N^2(t)\, c^2(t)\, dt^2 + a^2(t)\, d\Omega_3^2, \qquad c(t) = c_0 \left(\frac{a(t)}{a_0}\right)^\alpha$$

$$\Psi(a) = a^{-\frac{q+1}{2}} (\beta a)^{-3\alpha} \exp\left(-\frac{\beta a^4}{2}\right)$$



Gravity's Rainbow $\Longrightarrow$ Application to Inflation

[R. Garattini and M. Sakellariadou, Phys. Rev. D 90 (2014) 4, 043521; arXiv:1212.4987 [gr-qc]]

$$ds^2 = -\frac{N^2(t)}{g_1^2(E/Ep)}dt^2 + \frac{a^2(t)}{g_2^2(E/Ep)}d\Omega_3^2 \quad \Leftrightarrow \text{Distorted } \mathit{FLRW\ metric}$$

$$\left[16\pi G\frac{g_1^2(E/E_{\mathrm{Pl}})}{g_2^3(E/E_{\mathrm{Pl}})}\tilde{G}_{ijkl}\tilde{\pi}^{ij}\tilde{\pi}^{kl} - \frac{\sqrt{\tilde{g}}}{16\pi G g_2(E/E_{\mathrm{Pl}})}\left(\tilde{R} - \frac{2\Lambda}{g_2^2(E/E_{\mathrm{Pl}})}\right)\right]\Psi(a) = 0\,.$$

$$\left[-\frac{\partial^2}{\partial a^2} - \frac{q}{a}\frac{\partial^2}{\partial a} + \left(\frac{3\pi g_2(E/E_P)}{2Gg_1(E/E_P)}\right)^2 a^2\left(1 - \frac{\Lambda a^2}{3g_2^2(E/E_P)}\right)\right]\Psi(a) = 0$$

$$\left[-\frac{\partial^2}{\partial a^2} - \frac{q}{a}\frac{\partial^2}{\partial a} + \left(\frac{3\pi}{2G}\right)^2 a^2\left(1 - \frac{\Lambda_{eff}a^2}{3}\right)\right]\Psi(a) = 0 \qquad \Lambda_{eff} = \Lambda\left(1 + \frac{4G}{\Lambda\pi}V(\phi)\right)$$

$$\Psi_{\mathrm{V}}^{\mathrm{inside}}(a) \simeq \left(1 - \frac{a^2}{g_2^2(E/E_{\mathrm{Pl}})a_0^2}\right)^{-\frac{1}{4}} \exp\left[-\frac{2}{3}z_0^{\frac{3}{2}}(E/E_{\mathrm{Pl}})\left\{1 - \left(1 - \frac{a^2}{g_2^2(E/E_{\mathrm{Pl}})a_0^2}\right)^{\frac{3}{2}}\right\}\right],$$

$$z_0(E/E_{\mathrm{Pl}}) = \left[\frac{3\pi a_0^2 g_2^3(E/E_{\mathrm{Pl}})}{4Gg_1(E/E_{\mathrm{Pl}})}\right]^{\frac{2}{3}}.$$

$$\Psi_{\mathrm{HH}}^{\mathrm{inside}}(a) \simeq \left(1 - \frac{a^2}{g_2^2(E/E_{\mathrm{Pl}})a_0^2}\right)^{-\frac{1}{4}} \exp\left[\frac{2}{3}z_0^{\frac{3}{2}}(E/E_{\mathrm{Pl}})\left\{1 - \left(1 - \frac{a^2}{g_2^2(E/E_{\mathrm{Pl}})a_0^2}\right)^{\frac{3}{2}}\right\}\right],$$

$$a_0^2 = \frac{3}{\Lambda}$$

Generalization

From Mini-SuperSpace to Field Theory in 3+1 Dimensions

The Cosmological Constant as a Zero Point Energy Calculation

$$\frac{1}{V}\frac{\int D\mu[h]\Psi^*[h]\int_\Sigma d^3x\widehat{\Lambda}_\Sigma\Psi[h]}{\int D\mu[h]\Psi^*[h]\Psi[h]}=-\frac{\Lambda}{\kappa}$$

Induced Cosmological "Constant"

$$D\mu[h]=D\left[h_{ij}^{\perp}\right]D\left[\xi_j^T\right]D[h]J$$

Solve this infinite dimensional PDE with a Variational Approach without matter fields contribution

Ψ is a trial wave functional of the gaussian type

Schrödinger Picture

Spectrum of Λ depending on the metric

Energy (Density) Levels

Eliminating Divergences using Gravity's Rainbow

[R.G. and G.Mandanici, Phys. Rev. D 83, 084021 (2011), arXiv:1102.3803 [gr-qc]]

One loop Graviton Contribution

$$\begin{cases} m_1^2(r) = \dfrac{6}{r^2}\left(1-\dfrac{b(r)}{r}\right)+\dfrac{3b'(r)}{2r^2}-\dfrac{3b(r)}{2r^3} \\ m_2^2(r) = \dfrac{6}{r^2}\left(1-\dfrac{b(r)}{r}\right)+\dfrac{b'(r)}{2r^2}+\dfrac{3b(r)}{2r^3} \end{cases}$$

We can define an r-dependent radial wave number

$$k^2(r,l,E_{nl}) = \frac{E_{nl}^2}{g_2^2(E/E_P)} - \frac{l(l+1)}{r^2} - m_i^2(r) \qquad r \equiv r(x)$$

Minkowski - de Sitter - Anti-de Sitter

$$m_1^2(r) = m_2^2(r) = m_0^2(r) \rightarrow x = \sqrt{m_0^2(r)/E_P^2}$$

$$\frac{\Lambda}{8\pi G} = -\frac{1}{3\pi^2}\sum_{i=1}^{2}\int_{E^*}^{+\infty} E_i g_1(E/E_P) g_2(E/E_P) \frac{d}{dE_i}\sqrt{\left(\frac{E_i^2}{g_2^2(E/E_P)} - m_i^2(r)\right)^3}\, dE_i$$

Standard Regularization

$$\frac{\Lambda}{8\pi G} = -\frac{1}{16\pi^2}\int_{\sqrt{m_i^2(r)}}^{+\infty} \frac{\omega_i^2}{\left(\omega_i^2 - m_i^2(r)\right)^{\varepsilon-\frac{1}{2}}} d\omega_i$$

Popular Choice...... $\rightarrow$ Not Promising

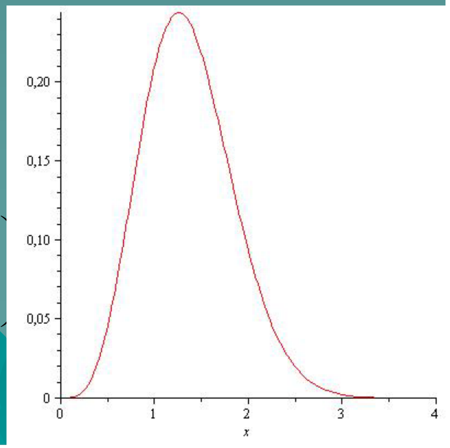
$$g_1(E/E_P) = 1-\eta\left(\frac{E}{E_P}\right)^n$$

$$g_2(E/E_P) = 1$$

Failure of Convergence

$$g_1(E/E_P) = \exp\left(-\alpha\frac{E^2}{E_P^2}\right)\left(1+\beta\frac{E}{E_P}\right.$$

$$g_2(E/E_P) = 1$$



Conclusions and Outlooks

- The WDW equation can be considered as a Sturm-Liouville Problem→ Rayleigh-Ritz Variational procedure.
- In ordinary GR, the MSS approach does not produce a cosmological constant.
- In ordinary GR, we need a cut-off or a regularization/renormalization scheme.
- A distorted GR (Gravity's Rainbow, VSL Cosmology, G.U.P.) produces a non trivial cosmological constant as an eigenvalue of a Rayleigh-Ritz variational procedure.
- In distorted GR, for example in Gravity's Rainbow we can compute divergent quantum observables without a Regularization/Renormalization procedure. This also happens in NonCommutative geometries. We have a tool for ZPE Computation.
- A connection between Horava-Lifshitz theory without detailed balanced condition and with projectability and Gravity's Rainbow seems possible, at least in a FLRW metric. (It works also for a SSM!!!)
- The WDW equation in distorted GR can be used also when a Naked singularity is considered. Many other applications….

→ Next step…..Kerr