Spacetime-noncommutativity regime of Loop Quantum Gravity

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Introduction

We establish a bridge between two different quantum-gravity (QG) approaches:

- Loop Quantum Gravity (LQG) [1,2] non-perturbative (background independent) quantization of gravity, almost complete formalism, in-principle valid for all regimes, BUT very difficult to extract testable predictions!
- Spacetime Noncommutativity [3,4]: a way to characterize a non-classical spacetime, maybe a first step toward QG, confined to the Minkowski (flat) limit, BUT there is a phenomenology[5]!

Still no direct link between them!

[1] C. Rovelli, Living Rev. Rel. 1, (1998) 1.

[2] A. Ashtekar and J. Lewandowski, Class. Quant. Grav. 21, R53-R152 (2004).

[3] H. S. Snyder, Phys. Rev. 71, 38 (1947)

[4] S. Doplicher, K. Fredenhagen, J. E. Roberts, Phys. Lett. B331, 39 (1994)

[5] G. Amelino-Camelia, Living Rev. Rel. 16, (2013) 5.

Motivation

No clear link between Spacetime Noncommutativity and QG:

- Just heuristic arguments but no rigorous derivation
- Some hints from Strings[6] and 2+1 gravity[7]
- What about LQG?

Minkowski regime of LQG poorly understood:

- Quantum spacetime? Noncommutative?
- Fate of Lorentz symmetries? Exact, broken or deformed?
- Experimental tests?

To answer such questions we must look at the SYMMETRIES: how are they encoded?

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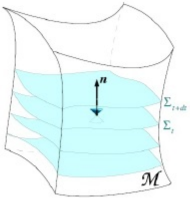
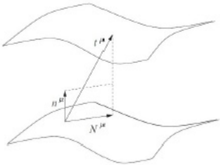
[6] N. Seiberg, E. Witten, JHEP 09, 032 (1999).

[7] L. Freidel, E. R. Livine, Phys. Rev. Lett. 96, 221301 (2006).

Hamiltonian formulation of General Relativity (ADM)

..let us start from the classical theory!
Based on the 3+1 foliation of spacetime[8]





- $g_{\mu\nu}(x) \leftrightarrow h_{ij}(x), N^k(x), N(x)$
- Phase-space variables:
  $\{\pi^{ij}(x), h_{kl}(y)\} = -\frac{1}{2}(\delta^i_k \delta^j_l + \delta^i_l \delta^j_k)\delta^{(3)}(x-y)$

Does it break general covariance?..of course NOT!

[8] R. L. Arnowitt, S. Deser, C. W. Misner, Gen. Rel. Grav. 40, 1997 (2008).

Constraints

Diffeomorphism invariance is implemented by means of constraints:

$$H[N] = \int d^3x \quad N(x)(\frac{\pi_{lk}\pi^{lk}}{\sqrt{-h}} - \frac{\pi^2}{2\sqrt{-h}} -^3 R\sqrt{-h})$$
$$D[N^k] = -2\int d^3x \quad N^k(x)h_{kj}(x)D_l\pi^{lj}(x)$$

which close the hypersurface-deformation algebra (HDA) [9]:

$$\{D[N^i], D[N'^j]\} = D[N'^j\partial_j N^i - N^j\partial_j N'^i]$$
$$\{D[N^i], H[N']\} = H[N^j\partial_j N']$$
$$\{H[N], H[N']\} = D[h^{ij}(N\partial_j N' - N'\partial_j N)]$$

the HDA encodes general covariance!

[9] P. A.M. Dirac, Proc. Roy. Soc. Lond. A246, 333 (1958).

Minkowski limit: Poincaré algebra from linear hypersurface deformations

If we restrict to linear coordinate changes:

$$N^k(x) = \Delta x^k + R^k_i x^i \quad N(x) = \Delta t + v_i x^i \quad h_{ij} = \delta_{ij}$$

the HDA reduces to the Poincaré algebra[10]:

$$\{P_\mu, P_\nu\} = 0 \quad \{M_{\mu\nu}, P_\rho\} = \eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu$$
$$\{M_{\mu\nu}, M_{\rho\sigma}\} = \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}$$

In particular we have:

$$\begin{aligned} \{D[N^i], D[N^j]\} &\longrightarrow \{P_i, P_j\}, \quad \{R_i, P_j\}, \quad \{R_i, R_j\} \\ \{H[N], D[N^j]\} &\longrightarrow \{P_i, P_0\}, \quad \{R_i, P_0\}, \quad \{R_i, B_j\}, \quad \{P_i, B_j\} \\ \{H[N], H[N']\} &\longrightarrow \{B_i, P_0\}, \quad \{B_i, B_j\} \end{aligned} \tag{1}$$

[10] T. Regge, C. Teitelboim, Annals Phys. 88, 286 (1974).
M. Bojowald, Canonical Gravity and Applications (2010).

Loop quantization

First step is Ashtekar's formulation[11] (use $SU(2)$ fields!):

$$(h_{ij}, \pi^{ij}) \longrightarrow (A_i^a, E_a^i)$$

where:

$$A_i^a = \Gamma_i^a + \gamma K_i^a$$
$$E_a^i = \sqrt{det(h)} e_a^i, \quad h^{ij} = e_a^i e_b^j \delta_{ab}$$

To quantize the theory one must turn to[12]

1. *Holonomies*: $h_e(A) = exp(\int_e ds A_i^a \dot{e}^i \tau_a) \Rightarrow$ holonomy corrections!
2. *Fluxes*: $F_S(E) = \int_S d^2 y n_i E_a^i \Rightarrow$ inverse-triad corrections!

$\hookrightarrow$ Do they affect the spacetime structure?

[11] A. Ashtekar, Phys. Rev. Lett. 57, 2244 (1986).

[12] C. Rovelli and L. Smolin, Nucl. Phys. B331, 80 (1990).

Deformed hypersurface deformations: spherical symmetry

Introduce rotationally invariant densitized triads:

$$E = E^i_a \tau^a \frac{\partial}{\partial x^i} = E^r(r)\tau_3 \sin\theta \frac{\partial}{\partial r} + E^\phi(r)\tau_1 \sin\theta \frac{\partial}{\partial \theta} + E^\phi(r)\tau_2 \frac{\partial}{\partial \phi},$$

which are conjugate to:

$$K = K^a_i \tau_a dx^i = K_r(r)\tau_3 dr + K_\phi(r)\tau_1 d\theta + K_\phi(r)\tau_2 \sin\theta d\phi$$

Considering only the point-wise holonomy of the homogeneous connections:

$$K_\phi \to \frac{\sin(\delta K_\phi)}{\delta}$$

one finds:

$$\{H[N], H[N']\} = D[\cos(2\delta K_\phi)\frac{E^r}{(E^\phi)^2}(N\partial_r N' - N'\partial_r N)] \qquad (2)$$

Covariance is NOT violated, BUT deformed! $\to$ correspondingly deformed relativistic symmetries? consequences for spacetime?

Deformed Poincaré algebra: a bridge to noncommutativity?

Restricting to linear hypersurface deformations[13]:

$$[B_r, P_0] = iP_r \cos(\lambda P_r) \quad [B_r, P_r] = iP_0 \quad [P_r, P_0] = 0$$

thanks to the relation $\lambda P_r = -\frac{1}{G}\frac{K_\phi}{\sqrt{E^r}} = 2\delta K_\phi$ ($\lambda \approx 10^{-35}m$) between K_ϕ and the Brown-York (ADM) momentum:

$$P = 2\int_{\partial\Sigma} d^2 z \upsilon_i (n_j \pi^{ji} - \overline{n}_j \overline{\pi}^{ji})$$

The Minkowski limit of the HDA with quantum corrections produces a deformation of the Poincaré algebra!
⇒ Deformed Special Relativity[14] (DSR) derived from LQG!
⇒ Which is the underlying quantum spacetime?

[13] M. Bojowald, G. M. Paily, Phys. Rev. D87, 044044 (2013).
[14] G. Amelino-Camelia, Int. J. Mod. Phys. D11, 35 (2002).

Proposal: κ-Minkowski

κ-Minkowski noncommutative spacetime[15] is defined by:

$$[\widehat{X}_r, \widehat{X}_0] = -i\lambda\widehat{X}_r$$

To prove that a deformed Poincaré algebra generates its symmetries we need to:

- check the fulfilment of all the Jacobi identities
- compute the coproducts: ΔB_r, ΔP_r and ΔP_0

Non-linear deformations of the Poincaré algebra are Hopf (rather than Lie) algebraic structures!
WARNING: violation of Leibniz's rule:
$G \triangleright (fg) \neq (G \triangleright f)g + f(G \triangleright g)$ which must involve only the generators of the Hopf algebra: $\{B_r, P_r, P_0\}$

[15] G. Amelino-Camelia and S. Majid, Int. J. Mod. Phys. A15, 4301 (2000).

Representations

We propose a general *ansatz* for the representations:

$$B_r = F(p_0, p_r)\widehat{X}_r p_0 - G(p_0, p_r)\widehat{X}_0 p_r, \quad P_r = Z(p_r), \quad P_0 = p_0$$

where p_r, p_0 are standard momenta that act on κ-Minkowski coordinates as follows:

$$[p_r, \widehat{X}_0] = i\lambda p_r, \quad [p_r, \widehat{X}_r] = -i, \quad [p_0, \widehat{X}_0] = i, \quad [p_0, \widehat{X}_r] = 0$$

We find that $F(p_0, p_r), G(p_0, p_r), Z(p_r)$ must obey:

$$\lambda Z(p_r)\sin(\lambda Z(p_r)) + \cos(\lambda Z(p_r)) = \frac{\lambda^2 {p_r}^2}{2} + 1,$$

$$F(p_0, p_r) = G(p_r)e^{\lambda p_0} = \frac{Z(p_r)\cos(\lambda Z(p_r))e^{\lambda p_0}}{p_r},$$

$$G(p_r) = \frac{Z(p_r)\cos(\lambda Z(p_r))}{p_r}$$

Jacobi identities

Then we verify that all the Jacobi identities involving $\{\widehat{X}_r, \widehat{X}_0, B_r, P_r, P_0\}$ are satisfied.
By a way of example we have that:

$$[[B_r, \widehat{X}_r], \widehat{X}_0] + [[\widehat{X}_0, B_r], \widehat{X}_r] + [[\widehat{X}_r, \widehat{X}_0], B_r] =$$
$$-i[\frac{(Z' \cos(\lambda Z) - \lambda Z Z' \sin(\lambda Z))p_r - Z\cos(\lambda Z)}{p_r^2} x_r p_0, \widehat{X}_0] +$$
$$+i[\frac{(Z' \cos(\lambda Z) - \lambda Z Z' \sin(\lambda Z))p_r - Z\cos(\lambda Z)}{p_r^2} x_0 p_r, \widehat{X}_0] +$$
$$-[i\frac{Z\cos(\lambda Z)}{p_r} x_r - \lambda[B_r, \widehat{X}_r]p_r - i\lambda\frac{Z\cos(\lambda Z)}{p_r} x_r p_0, \widehat{X}_r] +$$
$$+[i\frac{Z\cos(\lambda Z)}{p_r} x_0, \widehat{X}_0] + i\lambda[B_r, \widehat{X}_r] = 0$$

Perturbative explicit solution

We are not able to solve analytically the equation for $Z(p_r)$:

$$\lambda Z(p_r)\sin(\lambda Z(p_r)) + \cos(\lambda Z(p_r)) = \frac{\lambda^2 {p_r}^2}{2} + 1,$$

Thus we provide a perturbative solution up to the quartic order:

$$Z(p_r) \simeq p_r + \frac{1}{8}\lambda^2 {p_r}^3 + \frac{55}{1152}\lambda^4 {p_r}^5 + \bigcirc(\lambda^5)$$

This allow us to calculate also the coproducts of the generators!

Coproducts

A way to compute coproducts is to act with generators on the product of two plane waves:

$$G \triangleright (e^{ik_r \widehat{X}_r} e^{ik_0 \widehat{X}_0} e^{iq_r \widehat{X}_r} e^{iq_0 \widehat{X}_0}) = G \triangleright (e^{i(k_r + e^{-\lambda k_0} q_r)\widehat{X}_r} e^{i(k_0+q_0)\widehat{X}_0})$$

We find for the boost:

$$\begin{aligned}
\Delta B_r =\ & B_r \otimes \mathbf{1} + \mathbf{1} \otimes B_r - \lambda P_0 \otimes B_r + \frac{1}{8}\lambda^2 {P_r}^2 \otimes B_r + \frac{1}{2}\lambda^2 {P_0}^2 \otimes B_r - \frac{3}{8}\lambda^2 B_r \otimes {P_r}^2 - \frac{3}{4}\lambda^2 P_r B_r \otimes P_r \\
& - \frac{3}{4}\lambda^2 P_r \otimes P_r B_r - \frac{5}{8}\lambda^3 P_0 {P_r}^2 \otimes B_r + \frac{3}{4}\lambda^3 P_0 P_r \otimes P_r B_r - \frac{3}{4}\lambda^3 {P_r}^2 B_r \otimes P_0 - \frac{3}{4}\lambda^3 {P_r}^2 \otimes P_0 B_r \\
& - \frac{3}{4}\lambda^3 P_r B_r \otimes P_0 P_r - \frac{3}{4}\lambda^3 P_r \otimes P_0 P_r B_r + \frac{67}{1152}\lambda^4 P_r^4 \otimes B_r + \frac{15}{64}\lambda^4 {P_r}^2 \otimes {P_r}^2 B_r - \frac{1}{8}\lambda^4 {P_0}^4 \otimes B_r \\
& + \frac{9}{16}\lambda^4 {P_0}^2 {P_r}^2 \otimes B_r + \frac{15}{64}\lambda^4 {P_r}^2 B_r \otimes {P_r}^2 - \frac{167}{288}\lambda^4 {P_r}^3 B_r \otimes P_r - \frac{59}{288}\lambda^4 P_r \otimes {P_r}^3 B_r \\
& - \frac{97}{144}\lambda^4 {P_r}^3 \otimes P_r B_r - \frac{3}{8}\lambda^4 {P_0}^2 P_r \otimes P_r B_r + \frac{3}{4}\lambda^4 P_0 {P_r}^2 \otimes P_0 B_r + \frac{3}{4}\lambda^4 P_0 P_r \otimes P_0 P_r B_r \\
& + \frac{11}{144}\lambda^4 P_r B_r \otimes {P_r}^3 - \frac{3}{4}\lambda^4 {P_r}^2 B_r \otimes {P_0}^2 - \frac{3}{4}\lambda^4 {P_r}^2 \otimes {P_0}^2 B_r - \frac{5}{1152}\lambda^4 B_r \otimes {P_r}^4 \\
& - \frac{3}{8}\lambda^4 P_r B_r \otimes {P_0}^2 P_r - \frac{3}{8}\lambda^4 P_r \otimes {P_0}^2 P_r B_r
\end{aligned}$$

Coproducts

The coproduct of the momentum is:

$$\begin{aligned}\Delta P_r = {} & P_r \otimes 1 + 1 \otimes P_r + \lambda P_r \otimes P_0 + \frac{1}{2}\lambda^2 P_r \otimes P_0{}^2 - \frac{1}{8}\lambda^2 Pr \otimes P_r{}^2 + \frac{3}{8}\lambda^2 P_r{}^2 \otimes P_r + \frac{1}{4}\lambda^3 P_r{}^3 \otimes P_0 \\ & + \frac{3}{8}\lambda^3 P_r \otimes P_0 P_r{}^2 + \frac{3}{4}\lambda^3 P_r{}^2 \otimes P_0 P_r + \frac{1}{2}\lambda^4 P_r{}^3 \otimes P_0{}^2 - \frac{49}{1152}\lambda^4 P_r \otimes P_r{}^4 + \frac{11}{36}\lambda^4 P_r{}^3 \otimes P_r{}^2 \\ & - \frac{1}{8}\lambda^4 P_r \otimes P_0{}^4 + \frac{7}{16}\lambda^4 P_r \otimes P_0{}^2 P_r{}^2 + \frac{1}{18}\lambda^4 P_r{}^2 \otimes P_r{}^3 + \frac{167}{1152}\lambda^4 P_r{}^4 \otimes P_r + \frac{3}{4}\lambda^4 P_r{}^2 \otimes P_0{}^2 P_r\end{aligned}$$

The coproduct of the energy is:

$$\begin{aligned}\Delta P_0 = {} & P_0 \otimes 1 + 1 \otimes P_0 + \lambda P_r \otimes P_r + \frac{1}{2}\lambda^2 P_0 \otimes P_0{}^2 + \frac{1}{2}\lambda^2 P_0{}^2 \otimes P_0 - \frac{1}{2}\lambda^2 P_0 \otimes P_r{}^2 - \lambda^2 P_0 P_r \otimes P_r \\ & + \frac{1}{2}\lambda^2 P_r{}^2 \otimes P_0 - \frac{1}{8}\lambda^3 P_r \otimes P_r{}^3 + \frac{3}{8}\lambda^3 P_r{}^3 \otimes P_r + \frac{1}{2}\lambda^3 P_0{}^2 P_r \otimes P_r - \lambda^3 P_0 P_r{}^2 \otimes P_0 + \frac{1}{4}\lambda^4 P_r{}^4 \otimes P_0 \\ & - \frac{1}{8}\lambda^4 P_0 \otimes P_0{}^4 - \frac{1}{8}\lambda^4 P_0{}^4 \otimes P_0 + \frac{1}{8}\lambda^4 P_0 P_r \otimes P_r{}^3 + \frac{1}{4}\lambda^4 P_0 \otimes P_0{}^2 P_r{}^2 \\ & + \frac{3}{4}\lambda^4 P_0{}^2 P_r{}^2 \otimes P_0 - \frac{7}{8}\lambda^4 P_0 P_r{}^3 \otimes P_r\end{aligned}$$

B_r, P_r, P_0 generate the relativistic symmetries of κ-Minkowski!

κ-Poincaré: bicrossproduct basis

Most used basis[16,17] defined as:

$$[M_r, Q_r] = i\frac{1 - e^{-2\lambda Q_0}}{2\lambda} - i\frac{\lambda}{2}Q_r^2 \quad [M_r, Q_0] = iQ_r \quad [Q_0, Q_r] = 0$$

with coproducts given by:

$$\begin{aligned}
\Delta M_r &= M_r \otimes 1 + e^{-\lambda Q_0} \otimes M_r \\
\Delta Q_0 &= Q_0 \otimes 1 + 1 \otimes Q_0 \\
\Delta Q_r &= Q_r \otimes 1 + e^{-\lambda Q_0} \otimes Q_r
\end{aligned}$$

The generators $\{B_r, P_r, P_0\}$ satisfy different commutation relations!
$\Rightarrow$ However different bases of the SAME Hopf algebra can be related through NON-LINEAR maps! If so, this guarantees they act on the SAME noncommutative spacetime!

[16] S. Majid, H. Ruegg, Phys. Lett. B334 (1994) 348-354.

[17] J. Lukierski, H. Ruegg and W.J. Zakrzewski, Ann. Phys. 243 (1995) 90.

New basis: compatibility with κ-Minkowski

The desired map from $\{M_r, Q_r, Q_0\}$ to $\{B_r, P_r, P_0\}$ is simply:

$$B_r = \frac{Z(Q_r e^{\lambda Q_0}) \cos \lambda Z(Q_r e^{\lambda Q_0})}{Q_r e^{\lambda Q_0}} M_r \quad P_r = Z(\lambda Q_r e^{\lambda Q_0})$$

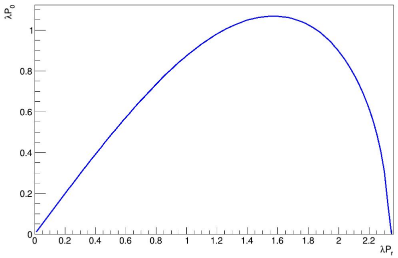
$$P_0 = \frac{\sinh(\lambda Q_0)}{\lambda} + \frac{\lambda}{2} Q_r^2 e^{\lambda Q_0}$$

where:

$$\lambda Z(\lambda Q_r e^{\lambda Q_0}) \sin(\lambda Z(\lambda Q_r e^{\lambda Q_0})) + \cos(\lambda Z(\lambda Q_r e^{\lambda Q_0})) = \frac{\lambda^2 Q_r^2 e^{2\lambda Q_0}}{2} + 1$$

$\Rightarrow$ COMPATIBLE WITH κ-MINKOWSKI!!

Modified dispersion relation



Experimentally interesting to look at the mass Casimir:

$$P_0^2 = 2\left(\frac{\lambda P_r \sin \lambda P_r + \cos \lambda P_r - 1}{\lambda^2}\right)$$

Non-monotonic function $\rightarrow$ $P_r^{max} = \frac{\pi}{2\lambda}$ maybe a cutoff! $\rightarrow$ Does it spoil relativistic invariance?

Conclusions and outlook

Achievements:

- Further light on the flat-spacetime (Minkowski) limit of LQG
- κ-Minkowski turns out to be compatible with LQG
- Spacetime noncommutativity finds support in a full-fledged QG approach
- Constrain LQG with tests of the MDR (GRBs!)

Goals:

- Find evidence of coproduct structures in LQG
- Combine holonomy and inverse-triad quantum corrections
- Extend the analysis to the full theory
- Study the de Sitter limit (q-de Sitter Hopf algebra?)
- Explore the FRW regime with the aim to derive the MDR in cosmology
- Compute deformations of the HDA in other QG approaches

THANKS FOR YOUR ATTENTION!